Quintic Hermite Splines

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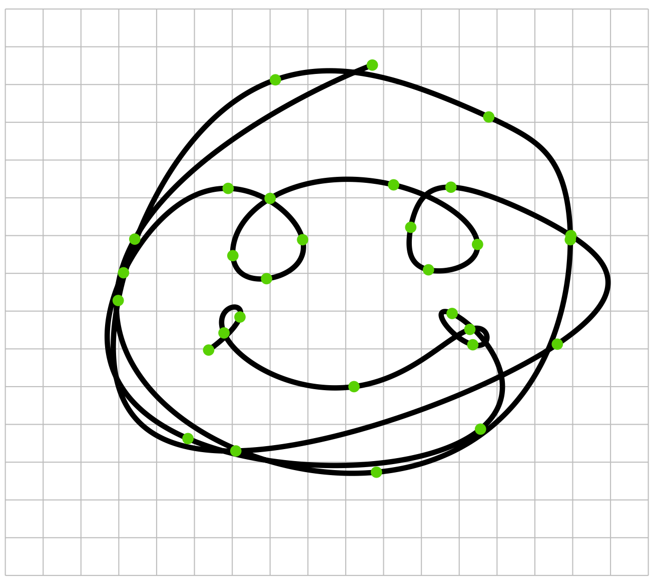

Figure 1: An example of a smooth spline.

The Objective

Given $N+1$ points $\vec{\boldsymbol{x}}_0, \ldots, \vec{\boldsymbol{x}}_N$, we want to find a smooth curve that goes through all points. In this paper we take a look at quintic hermite splines as a mean to fit these points. Quintic hermite splines are made up of N polynomials of order 5. Each polynomial k connects the points $\vec{\boldsymbol{x}}_k$ and $\vec{\boldsymbol{x}}_{k+1}$. As each dimension is treated independently, we will only consider a single dimension in the following, meaning points x_k are scalars. By fitting a spline for every dimension, 2- or 3-dimensional splines can be obtained.

On a Search for Coefficients

The quintic hermite spline is characterized by 5th order polynomials $S_k : [0,1] \rightarrow \mathbb{R}$ with $k \in \{0, \ldots, N\}$. As points are interpolated, each polynomial is only defined for the range $\tau = 0$ (start point) to $\tau = 1$ (end point). Its first four derivatives are easy to compute.

$$S(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4 + c_5\tau^5 \tag{1}$$

$$S^{(1)}(\tau) = c_1 + 2c_2\tau + 3c_3\tau^2 + 4c_4\tau^3 + 5c_5\tau^4 \tag{2}$$

$$S^{(2)}(\tau) = 2c_2 + 6c_3\tau + 12c_4\tau^2 + 20c_5\tau^3 \tag{3}$$

$$S^{(3)}(\tau) = 6c_3 + 24c_4\tau + 60c_5\tau^2$$

$$S^{(4)}(\tau) = 24c_4 + 120c_5\tau$$

Start and end point are set equal to the points we want to fit: $S_k(\tau = 0) = x_k$ and $S_k(\tau = 1) = x_{k+1}$. This implies that $S_k(1) = S_{k+1}(0)$. In matrix vector notation this leads to the following equation for start and end point.

$$\begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \\ x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

To obtain the coefficients, we simply solve the linear system.

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -10 & -6 & -\frac{3}{2} & 10 & -4 & \frac{1}{2} \\ 15 & 8 & \frac{3}{2} & -15 & 7 & -1 \\ -6 & -3 & -\frac{1}{2} & 6 & -3 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \\ x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \end{bmatrix}. \tag{4}$$

With the resulting formula from (4) we can calculate parameters of the connecting polynomials given a start and end point and its 1st and 2nd order slope. Thus, the only remaining unknowns are the slopes $\dot{x}_k$ and $\ddot{x}_k$ for intermediate points $k \in \{1, \ldots, N-1\}$.

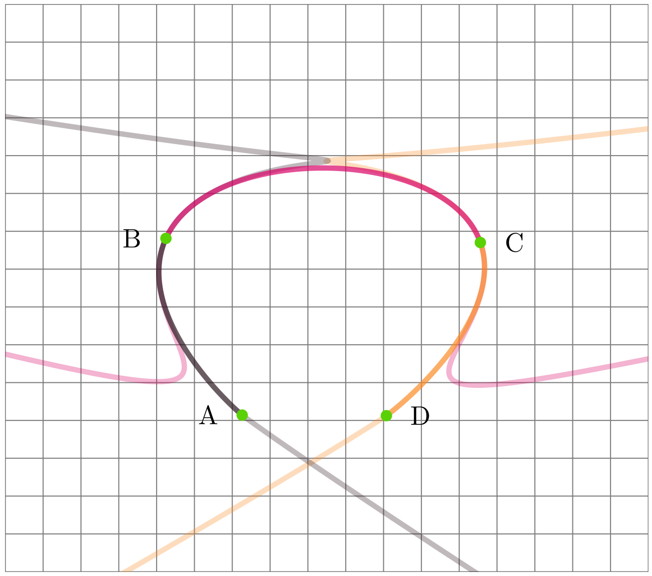

Figure 2: Polynomials of a spline fitted through points A, B, C, and D. You can observe the undefined behaviour of each polynomial outside of its defined range.

Solving Slopes

We can solve for these slopes by enforcing constraints. A reasonable constraint is to require continuity in C^4. Therefore we require the fourth derivative to be continuous, which is the case if the derivative of one polynomial is equal to that of the next at the same point in space. Since $S_k(1)$ and $S_{k+1}(0)$ are same points, we get

$$\begin{aligned} & S_k^{(4)}(1) = S_{k+1}^{(4)}(0) \\ \Leftrightarrow \quad & 24c_{k,4} + 120c_{k,5} = 24c_{k+1,4} \\ \Leftrightarrow \quad & c_{k,4} + 5c_{k,5} = c_{k+1,4} \end{aligned} \tag{5}$$

And,

$$\begin{aligned} & S_k^{(3)}(1) = S_{k+1}^{(3)}(0) \\ \Leftrightarrow \quad & 6c_{k,3} + 24c_{k,4} + 60c_{k,5} = 6c_{k+1,3} \\ \Leftrightarrow \quad & c_{k,3} + 4c_{k,4} + 10c_{k,5} = c_{k+1,3} \end{aligned} \tag{6}$$

We need two equations because we are solving for two unknowns. In order to express (5) and (6) in terms of $\dot{x}$ and $\ddot{x}$, we first of all solve (1), (2), and (3) for c_3, c_4, and c_5 using x, $\dot{x}$, and $\ddot{x}$ so that we can substitute.

To solve for c_3, c_4, and c_5, we set

$$\begin{aligned}
x_{k+1} - x_k &= S_k(1) - S_k(0) = c_{k,1} + c_{k,2} + c_{k,3} + c_{k,4} + c_{k,5} \\
\dot{x}_{k+1} - \dot{x}_k &= S_k^{(1)}(1) - S_k^{(1)}(0) = 2c_{k,2} + 3c_{k,3} + 4c_{k,4} + 5c_{k,5} \\
\ddot{x}_{k+1} - \ddot{x}_k &= S_k^{(2)}(1) - S_k^{(2)}(0) = 6c_{k,3} + 12c_{k,4} + 20c_{k,5}
\end{aligned}$$

As a result, we get

$$\begin{aligned}
c_{k,3} &= \frac{\ddot{x}_{k+1}}{2} - \frac{3\,\ddot{x}_k}{2} + 10\,x_{k+1} - 10\,x_k - 4\,\dot{x}_{k+1} - 6\,\dot{x}_k \\
c_{k,4} &= \frac{3\,\ddot{x}_k}{2} - \ddot{x}_{k+1} - 15\,x_{k+1} + 15\,x_k + 7\,\dot{x}_{k+1} + 8\,\dot{x}_k \\
c_{k,5} &= \frac{\ddot{x}_{k+1}}{2} - \frac{\ddot{x}_k}{2} + 6\,x_{k+1} - 6\,x_k - 3\,\dot{x}_{k+1} - 3\,\dot{x}_k
\end{aligned}$$

We plug the result into (5) and (6) to obtain the following equations

$$7\dot{x}_{k+2} + 16\dot{x}_{k+1} + 7\dot{x}_k - \ddot{x}_{k+2} + \ddot{x}_k = 15x_{k+2} - 15x_k \tag{7}$$

$$-8\dot{x}_{k+2} + 8\dot{x}_k + \ddot{x}_{k+2} - 6\ddot{x}_{k+1} + \ddot{x}_k = -20x_{k+2} + 40x_{k+1} - 20x_k \tag{8}$$

This is a linear system. As we have two unknowns, $\dot{x}$ and $\ddot{x}$, and two equations, we can solve it. To make it solvable using an algorithm (e.g. by QR-decomposition), we express it in terms of $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$. We start by subdividing $\boldsymbol{A}$ into $\boldsymbol{A_{11}}$, $\boldsymbol{A_{21}}$, $\boldsymbol{A_{12}}$, and $\boldsymbol{A_{22}}$.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_{11}} & \boldsymbol{A_{12}} \\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}} \end{bmatrix}$$

We can compare coefficients from (7) and (8).

$$\boldsymbol{A_{11}} = \begin{bmatrix}
16 & 7 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
7 & 16 & 7 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 7 & 16 & 7 & \dots & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \dots & 7 & 16 & 7 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 7 & 16 & 7 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 7 & 16
\end{bmatrix}
\qquad
\boldsymbol{A_{21}} = \begin{bmatrix}
0 & -8 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
8 & 0 & -8 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & -8 & \dots & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \dots & 8 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 8 & 0 & -8 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 8 & 0
\end{bmatrix}$$

$$\boldsymbol{A_{12}} = \begin{bmatrix}
0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \dots & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0
\end{bmatrix}
\qquad
\boldsymbol{A_{22}} = \begin{bmatrix}
-6 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
1 & -6 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 1 & -6 & 1 & \dots & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 0 & 0 & \dots & 1 & -6 & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & -6 & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -6
\end{bmatrix}$$

Table 1: Submatrices of $\boldsymbol{A}$. These are obtained by comparing the coefficients of (7) and (8) for different k.

Finally, we obtain the complete matrix system by determining $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{b}}$.

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 & \dots & \dot{x}_{n-1} & \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 & \ddot{x}_4 & \dots & \ddot{x}_{n-1} \end{bmatrix}^T$$

$$\vec{\boldsymbol{b}} = \begin{bmatrix} 15x_2 - 15x_0 - 7\dot{x}_0 - \ddot{x}_0 \\ 15\left(x_3 - x_1\right) \\ 15\left(x_4 - x_2\right) \\ 15\left(x_5 - x_3\right) \\ \vdots \\ 15\left(x_n - x_{n-2}\right) - 7\dot{x}_n + \ddot{x}_n \\ -20\left(x_2 - 2x_1 + x_0\right) - 8\dot{x}_0 - \ddot{x}_0 \\ -20\left(x_3 - 2x_2 + x_1\right) \\ -20\left(x_4 - 2x_3 + x_2\right) \\ -20\left(x_5 - 2x_4 + x_3\right) \\ \vdots \\ -20\left(x_n - 2x_{n-1} + x_{n-2}\right) + 8\dot{x}_n - \ddot{x}_n \end{bmatrix}$$

Using this linear system, both $\dot{x}_k$ and $\ddot{x}_k$ for every $k \in \{1, \dots, N-1\}$ can be found. The spline can therefore be calculated given points x_k for all k, and start and end slopes $\dot{x}_0$, $\ddot{x}_0$, $\dot{x}_N$, and $\ddot{x}_N$. The solution always exists and is unique because it can be shown for every N, that matrix $\boldsymbol{A}$ has $\det \boldsymbol{A} \neq 0$.

The Result

As can be seen in Figure 3, the resulting quintic hermite spline is smooth and continuous.

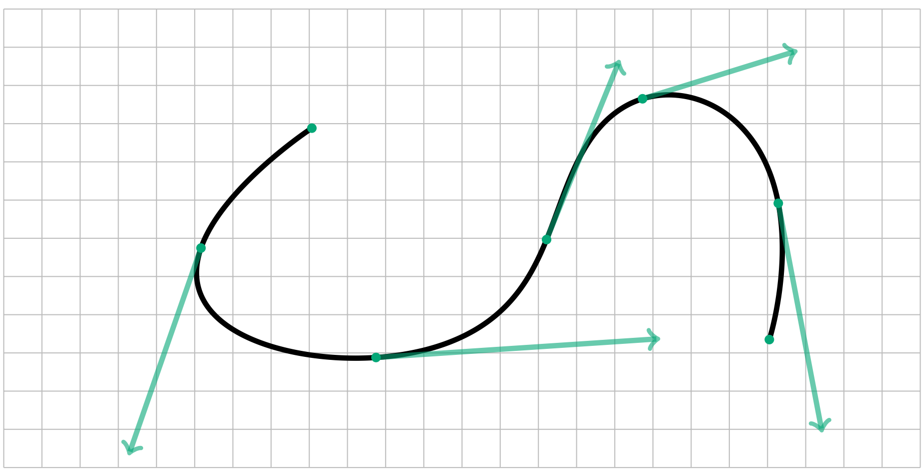

Figure 3: A quintic hermite spline calculated using this method. Green points are x_k. $\dot{x}_0$, $\ddot{x}_0$, $\dot{x}_N$, and $\ddot{x}_N$ were all set to zero. The tangents (1st derivatives) found by solving the linear system are highlighted.